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THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. IV.] JUNE 1912. [No. 3.

PROGRESS REPORT.

The following gentlemen have been elected members of the Society.—

(1) Mr. R. Anantaiyar, B.A., Pleader, Tirukkoilur.

(2) Mr. I. Srinivasa Aiyar, B.A., L.T., Principal, Hindu College, Tinnevelly.

(3) Mr. P. A. Subramania Iyer, B.A., L.T, Lecturer in Mathematics, Kottayam College, Kottayam.

(4) Mr. V. K. Aravamudhan, Student, Presidency College, 3, Swami Pillai Street, Triplicane, Madras (*concessional rate*).

(5) Mr. G. Ramachandran, Student, Presidency College, c/o Mr. T. V. Gopalaswami Aiyar, Treasury Dy. Collector, Bellary, (*concessional rate*).

(6) Sri Rajah Ankitham Venkata Jugga Row Bahadur Garu, F.R.A.S., F.R.C.I., F.R. Met. Soc., F.M.U., F.A.S, and Vice-President of the Astronomical Society of India, Zamindar of Sharemohamedpuram, Daba Gardens, Vizagapatam.

(7) Mr. Bhagat Ram, M.A., L.L.B., (Panjab), Senior Professor of Mathematics, Mohindra College, Patiala.

2. The following books have been purchased for the Library :—

(1) *Stability in Aviation*—by Dr. G. H. Bryan, Sc. D., 5*s*., (Macmillan & Co.)

(2) *Mathematical and Physical Papers, Vol. VI*—by Lord Kelvin, 10*s*., (Cambridge University Press).

(3) *Le Dernier Theoreme de Fermat*—by Professor E. Barbetta, 1*s*. 6*d*., (Gauthier-Villars, Paris).

(4) *Lecons sur les Hypotheses Cosmogonianes*—by Professor H. Poincare, 9*s*. 8*d*., (A. Herman & Fils, Paris).

(5) *The Proceedings of the London Mathematical Society*, Series 2, Vol. 10, £1-5*s*. (Published by Messrs. F. Hodgson & Sons, London.)

(6) *Principia Mathematica Vol. II*—by Messrs. Whitehead and Russell. £1-10*s*. (Published by the University Press, Cambridge.)

3 The Committee feel great pleasure to acknowledge with thanks the receipt of the following books from their respective publishers :—

(1) *Cours de Mathematiques Superieures*—a l'usage des candidats a la Licence es Sciences Physique, by Prof. L'abbe E. Stoffaes (Gauthier-Villars, Paris ; Tome I, 10 fr.

(2) „ „ Tome II, 10 fr.

(3) *Geometrie Rationnelle*, translated from the English edition of Dr. G. B. Halsted : by Prof. Paul Barbarin, (Gauthier Villars of Paris), 7 fr. 50.

(4) *A School Calculus*—by Messrs. A. M. McNeile and J. D. McNeile, (Mr. John Murray, London), 7*s*. 6*d*.

(5) *Elementary Trigonometry*—by W. E. Paterson, Oxford University Press, 3*s*.

(6) *School Algebra, Part I*—by do. do., 3*s*.

(7) *School Algebra, Part II*—by do. do., 3*s*.

(8) *Experimental and Theoretical Geometry*—by A. T. Warren, Oxford University Press, 3*s*.

(9) *Geometry for Schools*—by Borchardt and Perrott, Vols. I to III, (George Bell & Sons) 2*s*. 6*d*.

(10) *Proceedings* of the Edinburgh Mathematical Society, Vls. 28, 29.

(11) *Mathematical Notes*—Published by the Edinburgh Mathematical Society, Nos. 1 to 9.

(12) *Engineering Applications of Higher Mathematics*—by V. Karapetoff, Part I (Problems on Machine Design) Messrs. John Wiley & Sons, New York, 3*s*. net.

(13) *Differential and Integral Calculus*—by Love, Cambridge University Press, 5*s*.

(14) *Calculus for Beginners*—by Mercer, do. do. 6*s*.

(15) *A Course of Pure Geometry*—by Askwith, do. do. 5*s*.

(16) *Shorter Geometry*,—by Godfrey & Siddons University Press ... 2*s*.6*d*.

(17) *Solid Geometry*— do. do. do. do. ... 1*s*.6*d*.

(18) *Twenty-seven lines on the Cubic Surface*,—by Henderson, Mathematical Tracts, No. 13, do. do. ... 4*s*.6*d*.

(19) *Elementary Treatise on Cross Ratio Geometry*—by Milne, do. do. ... ... 6*s*.

(20) *Elementary Trigonometry*—by Swanwick, do. do. ... 4*s*.

(21) *Elementary Trigonometry*,—by Hobson and Jessop, do. do. ... ... 4*s*.6*d*.

(22) *Principia Mathematica, Vol. II*,—by Whitehead and Russell, do. do. ... ... 30*s*. net.

(23) *Lectures on the Differential Geometry of Curves and Surfaces*,—by Forsyth do. do. ... 21*s*. net.

(24) *Lecons sur les Principes de L'Analyse Vol. I.*—by Profesor R d'Adhemar—(Ganthier—Villars, Paris 1912) ... 10 fr.

(25) *Le Calcul des Probabilites et ses Applications*—by E. Carvallo—(Gauthier Villars, Paris 1912) ... ... 6·50 fr.

(26) *Mesure des Angles Hyperboles E'toilees et Developpante*—by Commandant D. Gartier (Gauthier—Villars, Paris, 1911) 2 fr.

POONA,
1st June, 1912.

D. D. KAPADIA,
Hony. Joint Scretary.

Some Properties of Curves and Surfaces.

By M. T. Naraniengar.

Let $P_1, P_2, \ldots P_n$ be a range of points and O a collinear point such that the algebraic sum of the products r together of $OP_1, OP_2, \ldots OP_n$ is zero. Then O may be called the *mean centre of the rth order* of the points $P_1, P_2, \ldots P_n$. Evidently there are r mean centres of the rth order. Also, since the sum of the products $(n-r)$ together of the reciprocals of $OP_1, OP_2, \ldots$ is zero, these points may be styled the harmonic centres of the $(n-r)^{th}$ order of the range. [For a discussion of the properties of these points see *Note on the Harmonic Centre*, p. 96 of this number].

1. *The locus of the mean centres of a system of parallel chords of a curve is a straight line.*

Let $\phi(\alpha, \beta, \gamma)=0$, be the trilinear equation of the curve, and (λ, μ, ν) the *direction sines* of the parallel chords. Then if $(\alpha', \beta', \gamma')$ be a point on one of the chords, the distance ρ of a point of intersection P of the chord and the curve from $(\alpha', \beta', \gamma')$ is given by

$$\phi\{(\alpha'+\lambda\rho), (\beta'+\mu\rho), (\gamma'+\nu\rho)\}=0 \quad \ldots \quad \ldots \quad (1)$$

If the point $(\alpha', \beta', \gamma')$ is the mean centre of the chord, the sum of the roots of (1) should be zero.

$$\therefore \qquad \alpha'\frac{d\phi}{d\lambda}+\beta'\frac{d\phi}{d\mu}+\gamma'\frac{d\phi}{d\nu}=0,$$

where ϕ stands for $\phi(\lambda, \mu, \nu)$.

Hence the locus of the mean centres of the parallel system is the straight line

$$\alpha\frac{d\phi}{d\lambda}+\beta\frac{d\phi}{d\mu}+\gamma\frac{d\phi}{d\nu}=0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

This line is evidently the $(n-1)^{th}$ polar of the point (λ, μ, ν), which is a point at infinity, with respect to the curve $\phi(\alpha, \beta, \gamma)$; and may be called the *diameter* of the parallel system. Cf. Salmon: Higher Plane Curves, § 127.

2. The Cartesian equation of the curve being written

$$u_n+u_{n-1}+u_{n-2}+\ldots\ldots=0,$$

the corresponding locus in Cartesian coordinates is

$$x\frac{d\phi_n}{dl}+y\frac{d\phi_n}{dm}+\phi_{n-1}=0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where ϕ_r denotes what u_r becomes when x, y are replaced by the *direction cosines* (l, m) of the parallel system.

3. *Definition*: The *centre* of a curve may be defined as that point through which if any chord be drawn the mean centre of the points of intersection is the point in question.*

4. If a curve has a centre, then all diameters pass through the centre. Conversely, all central chords are diameters.

If a diameter meets the curve at P, then P is the mean centre of the remaining $(n-1)$ points of intersection of the chord through P belonging to the parallel system.

5. The conditions for a centre may be obtained as follows :—

If (α, β, γ) be the ce. tre then the equation

$$\alpha\frac{d\phi}{d\lambda}+\beta\frac{d\phi}{d\mu}+\gamma\frac{d\phi}{d\nu}=0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

is true for any set of values of (λ, μ, ν) satisfying the fundamental equation

$$a\lambda+b\mu+c\nu=0.$$

In other words, $(a\lambda+b\mu+c\nu)$ must be a factor of (2). Hence, we may write down the identity†

$$\alpha\frac{d\phi}{d\lambda}+\ldots\equiv(a\lambda+b\mu+c\nu)(A\lambda^{n-2}+\ldots)$$

and equate corresponding coefficients on both sides. Eliminating (α, β, γ) from the resulting equations the required conditions are got.

Thus, it will be found that for a cubic *one* condition will do; for a quartic *two* conditions are necessary, and so on. Also, no curve can have more than one centre, as the equations in question are all *linear* in α, β, γ.

The required conditions may also be obtained from the Cartesian equation referred to the centre, which must be of the form

$$u_n+u_{n-2}+\ldots=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Examples:

(i) The centre of the cardioid $r=a(1-\cos\theta)$, or $(x^2+y^2)^2+2ax(x^2+y^2)=a^2y^2$, is the point $(-a/2, 0)$, which coincides with the mean centre of the points of contact of any three parallel tangents.

(ii) The centre of the limacon
$r=a+b\cos\theta$, or $(x^2+y^2-bx)^2=a^2(x^2+y^2)$
is the point $(-b/2, 0)$.

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* Chasles defines the centre of a curve as the centroid of the points of contact of parallel tangents, which is a fixed point. (*Vide*: Salmon; *Higher Plane Curves*, § 137).

† This may also be expressed by saying that the *first polar* of the centre should have the line at infinity as a factor.

(iii) The centre of the bicircular quartic $S^2+U=0$, is the centre of the circle S.

6. *If a curve has a centre, then all its asymptotes must pass through the centre.*

This is obvious from the Cartesian equation (4) referred to the centre.

In trilinear coordinates the conditions for an asymptote (λ, μ, ν) are that

$$\phi(\lambda, \mu, \nu)=0,$$

and
$$\alpha'\frac{d\phi}{d\lambda}+\beta'\frac{d\phi}{d\mu}+\gamma'\frac{d\phi}{d\nu}=0,$$

$(\alpha', \beta', \gamma')$ being a point on the asymptote.

That is, the equation of an asymptote is

$$\alpha\frac{d\phi}{d\lambda}+\dots=0,$$

where $\phi(\lambda, \mu, \nu)=0$. Hence by (2), the centre lies on the asymptote.

7. *If two curves are co-asymptotic, then they are co-diametral.*

This is evident from the Cartesian equation of a co-asymptotic system which have u_n, u_{n-1} common, and from (3).

8. *The locus of the mean centres of the second order of a system of parallel chords is a conic.*

Proceeding as in § 1, and equating the coefficient of ρ^2 to zero, we have for the required locus

$$\left(\alpha\frac{d}{d\lambda}+\beta\frac{d}{d\mu}+\gamma\frac{d}{d\nu}\right)^2\phi=0, \qquad \dots \quad \dots \quad (5)$$

which denotes a conic.

This locus is in fact the $(n-2)^{th}$ polar of the point (λ, μ, ν) at infinity with respect to the given curve.

9. *Generally, the locus of mean centres of the rth order of a system of parallel chords is a curve of the rth order, being the $(n-r)^{th}$ polar of (λ, μ, ν) with respect to the given curve.*

10. The mean centre of any chord whose direction sines are (λ, μ, ν) is also the mean centre of the same with respect to *every* polar curve of the point (λ, μ, ν) at infinity. In other words, the series of polar curves corresponding to the point (λ, μ, ν) at infinity have a common diameter, *viz.*, the $(n-1)^{th}$ polar of the point.

11. *The locus of the mean centre of the $(n-1)$ points of intersection of a chord parallel to an asymptote is a conic.*

For, if (λ, μ, ν) be the direction of the asymptote, so that $\phi(\lambda, \mu, \nu)=0$ the locus required is

$$\left(\alpha\frac{d}{d\lambda}+\beta\frac{d}{d\beta}+\gamma\frac{d}{d\lambda}\right)^2\phi=0.$$

which is a conic. This is the $(n-2)^{th}$ polar conic of (λ, μ, ν).

12. *Any point $(\alpha', \beta', \gamma')$ is the harmonic centre of the chord of the curve parallel to the $(n-1)^{th}$ polar of the point.*

For, the equation to the parallel chord is written

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho,$$

where
$$\lambda\frac{d\phi'}{d\alpha'}+\mu\frac{d\phi'}{d\beta'}+\nu\frac{d\phi'}{d\gamma'}=0,$$

ϕ' denoting $\phi(\alpha', \beta', \gamma')$. Now, the latter is the condition that the sum of the reciprocals of the roots of $\phi\{(\alpha'+\lambda\rho), ..\}=0$ should vanish, or that the harmonic centre of the chord should coincide with $(\alpha', \beta', \gamma')$.

13. Again, since the polar line of the point $(\alpha', \beta', \gamma')$ with respect to the first, second, ... polars is one and the same, it follows that: *the point $(\alpha', \beta', \gamma')$ is the harmonic centre of the parallel chord with respect to the first second,...polars of the point: it is thus the middle point of the parallel chord with respect to the polar conic.*

14. Corresponding to any point $(\alpha', \beta', \gamma')$ there are $(n-1)$ points on the polar line, whose $(n-1)^{th}$ polars are all parallel to one another. These are the $(n-1)$ harmonic centres of the polar line, and their properties follow from the Note on p. 96

15. *To draw a chord through a given point so as to have the point for the mean centre of the points of intersection of the chord.*

Take the first polar of the point, and draw a line through the point parallel to an asymptote of the first polar. This line will have the required property. The proof is obvious, and there are $(n-1)$ such lines.

16. If any chord through $(\alpha', \beta', \gamma')$ cuts the first, second, ... polars the distances of the points of cut are related thus:—

$\frac{1}{n}\Sigma$(reciprocals for original curve)

$=\frac{1}{n-1}\Sigma$(reciprocals for 1st polar)

$=\frac{1}{n-2}\Sigma$(reciprocals for 2nd polar)

......

=reciprocal of distance of polar line.

For, a line through $(\alpha', \beta', \gamma')$ cuts the first polar in points whose distances are given by

$$n\phi' + (n-1)\rho \left\{ \lambda\frac{d\phi'}{d\alpha'} + \mu\frac{d\phi'}{d\beta'} + \nu\frac{d\phi'}{d\gamma'} \right\}$$
$$+ (n-2)\frac{\rho^2}{\lfloor 2} \left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'} \right)^2 \phi' + \ldots = 0 \quad \ldots \quad (6)$$

The sum of the reciprocals of the roots is

$$-(n-1)\left(\lambda\frac{d\phi'}{d\alpha'} + \ldots \right) \div n\phi'$$

But the sum of the reciprocals of the roots of (1) § 1, is

$$-\left(\lambda\frac{d\phi'}{d\alpha'} + \ldots \right) \div \phi'$$

Hence, the first of the relations given above. Similarly, the other results follow. [*Cf.* Ed. *Dffl. Calc*, § 307, and Salmon §§ 133-4.]

17. The following further relations are deduced in a like manner:

$$\frac{Q_2}{n(n-1)} = \frac{{}_1Q_2}{(n-1)(n-2)} = \frac{{}_2Q_2}{(n-2)(n-3)} = \ldots\ldots$$

where ${}_rQ_2$ stands for the sum of the products two together of the reciprocals for the rth polar.

And, so on.

18. Consider the inverses of the points given by (1) with respect to $(\alpha', \beta', \gamma')$. The inverse points are determined by

$$\rho^n.\phi' + k^2.\rho^{n-1}\left(\lambda\frac{d}{d\alpha'} + \right)\phi' + \ldots = 0.$$

The harmonic centres of these are given by

$$n\rho^{n-1}\phi' + (n-1)k^2.\rho^{n-2}.\left(\lambda\frac{d}{d\alpha'} + \cdots \right)\phi' + \ldots = 0.$$

The inverses of the points given by § 16, (6) are also found from a similar equation.

Hence, we have the theorem:

"*The inverses of the points of cut of any line through* $(\alpha', \beta', \gamma')$ *and the first polar are the harmonic centres of the inverses of the intersections with the original curve.*

Similarly; the inverses of the points of cut with the second polar are the harmonic centres of the second order of the same set of points. And so on.

Lastly, the inverse of the point of cut with the $(n-1)^{th}$ *polar line is the "mean centre" of the same set of points.*

The above may also be proved easily by starting with the Cartesian equation

$$u_n + u_{n-1} + \ldots u_0 = 0,$$

and inverting, with respect to the origin, the points of intersection with the line $x/\cos\theta=y/\sin\theta=\rho$, &c.

19. *If two curves cut each other, the mean centre of the points of cut is the same as for the intersections of the asymptotes of the curves.*

For, if the curves be of the form

$$\phi\equiv u_n+u_{n-1}+u_{n-2}+\ldots=0,$$
$$\psi\equiv v_m+v_{m-1}+v_{m-2}+\ldots=0,$$

their mn intersections are got by solving these equations simultaneously.

Now, if y be eliminated, the coefficients of x^{mn} & x^{mn-1} in the resulting equation in x, will depend only upon u_n, u_{n-1}, v_m, v_{m-1}. Thus the *arithmetic mean* of the $m.n$ roots is the same as for the asymptotes of the curves.*

Hence, we have the following:

If one system of co-asymptotic curves intersects another system of co-asymptotic curves, the mean centre of the points of intersection is the same for all the members of the two systems.

20. To determine coordinates of the mean centre we may proceed thus:

Denote the asymptotes of ϕ & ψ by

$$y=\mu x+\nu \text{ and } y=\mu' x+\nu';$$

so that

$$f_n(\mu)=0,\ \nu=-\frac{f_{n-1}(\mu)}{f_n'(\mu)}$$

$$g_m(\mu')=0,\ \nu'=-\frac{g_{m-1}(\mu')}{g_m'(\mu')}$$

where f_r, g_r stand for u_r, v_r when $x=1$, $y=\mu$ or μ'.

Then $y=\mu'x+\nu'$ cuts $\phi=0$ in n points given by

$$x^n f_n\left(\mu'+\frac{\nu'}{x}\right)+x^{n-1}f_{n-1}\left(\mu'+\frac{\nu'}{x}\right)+\ldots=0,$$

i.e.,

$$x^n f_n(\mu')+x^{n-1}[\nu' f_n'(\mu')+f_{n-1}(\mu')]+\ldots\ldots=0.$$

Therefore, if X′ is the arithmetic mean of the roots

$$n.X'=\frac{f_n'(\mu').\ g_{m-1}(\mu')-f_{n-1}(\mu).\ g_m'(\mu')}{f_n(\mu').\ g_m'(\mu')}=F(\mu'),\text{ say.}$$

Hence, if ξ is the abicissa of the mean centre

$\xi=$ the arithmetic mean of the m values of X′, corresponding to different values of μ'.

$$=\frac{1}{m}.\ \Sigma(X')=\frac{1}{mn}\sum\left\{F(\mu')\right\}$$

* We may also argue thus: The coefficients of the highest degree terms depend upon the *forms* of the curves *at infinity*, and are therefore the same as for *the asymptotes* of the curves. Hence, &c.

Similarly, for the ordinate.

21. If a system of curves have the same terms of the nth, $(n-1)^{th}$, $(n-2)^{th}$ degrees, the first polar curve of (h,k) being

$$h\phi_x+k\phi_y+u_{n-1}+2u_{n-2}+\ldots=0$$

the mean centre of the intersections of the curve and the first polar will remain unchanged for the system.

Suppose that the equation $F_n=0$, denotes n straight lines, then the curve

$$F_n+F_{n-3}=0,$$

will be such that the mean centre of the points of contact of tangents from *any* point is the same whatever F_{n-3} may be ; being the same as for the $\frac{1}{2}n(n-1)$ points of intersection of the asymptotes.

22. Again, the mean centres of the *first* and *second* order of the points of intersection of any two curves, depend upon the three highest degree terms in each, and similar results may be deduced for systems of curves having these terms in common.

23. The properties mentioned in the previous articles are at once extended to space of three or more dimensions. Thus, if

$$\phi(\alpha, \beta, \gamma, \delta)=0,$$

denote a surface in three dimensions, we find that

(i) The locus of the mean centres corresponding to a system of parallel chords is

$$\sum\left(\alpha\frac{d\phi}{d\lambda}\right)=0,$$

which is a plane, being the $(n-1)^{th}$ polar of the point $(\lambda, \mu, \nu, \rho)$ at infinity.

(ii) Any two co-asymptotic surfaces are co-diametral ; and so on.

SHORT NOTES.

The Polar Circle.

1. Let the perpendiculars from A,B,C on the opposite sides of the $\triangle ABC$ be AD, BE, CF.

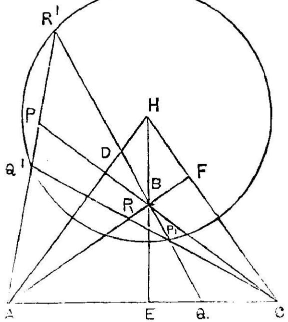
Draw angles DAP, EBQ, FCR in the same sense, each equal to $\theta = \sin^{-1} \rho/2R$, where ρ = radius of Polar Circle $= 2R\sqrt{-\cos A \cos B \cos C}$, so that, $\sin \theta = \sqrt{-\cos A \cos B \cos C}$.

Let P′Q′R′ be the triangle formed by AP, BQ, CR.

Since FCR or HCP′ $= \theta =$ EBQ or EBP′, a circle goes round HBP′C.

$\therefore$ angle Q′P′R′ or RP′B = BHC = A.

Similarly $\quad Q' = B$; $R' = C$.



Again $\quad \dfrac{HP'}{HC} = \dfrac{\sin HCP' \text{ or } HCR.}{\sin HBC} = \dfrac{\sin \theta.}{\sin (90^\circ + C).}$

$\therefore \quad HP' = 2R \cos C. \dfrac{\sin \theta.}{\cos C} = 2R \sin \theta = \rho.$

$\therefore \quad HP' = HQ' = HR'.$

$\therefore$ P′Q′R′ is incribed in the Polar Circle, so that its sides are $2\rho \sin A$, etc.

2. To prove that PQR is a straight line.

Since PBC is the polar of A, the range (AQ′PR) is harmonic.

$$\therefore \qquad \frac{Q'P}{PR'} = \frac{Q'A}{AR'}$$

And since ARBF is the polar of C, the range (CP′RQ′) is harmonic.

$$\therefore \qquad \frac{P'R}{RQ'} = \frac{P'C}{CQ'}$$

$$\therefore \qquad \frac{Q'P}{PR'} \cdot \frac{R'Q}{QP'} \cdot \frac{P'R}{RQ'} = \frac{Q'A}{AR'} \cdot \frac{R'Q}{QP'} \cdot \frac{P'C}{CQ'} \cdot$$

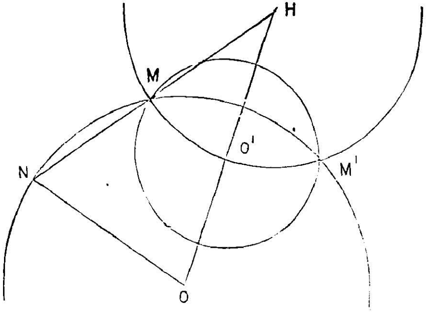
and the latter quantity $= -1$, being the product of the ratios of the segments of the sides of P′Q′R′ made by AQC.

$$\therefore \quad \frac{Q'P}{PR'} \cdot \frac{R'Q}{QP'} \cdot \frac{P'R}{RQ'} = -1, \text{ in the triangle } P'Q'R' ;$$

and therefore PRQ is a straight line.

The circumcircle O the Medial or N.P. circle O′, and the Polar circle H, have a common chord MM′ :

And H being the centre of similitude for circles O and O′, it follows that, if HM cut the circle O again in N, then $MN = HM = \rho = 2R \sin \theta$



$\therefore \quad OMN = \frac{1}{2}\pi - \theta.$

Also O′M is parallel to ON,

$\therefore \quad OMO' = MON = 2\theta.$

Hence, the circles O and H cut at an acute angle $\frac{1}{2}\pi - \theta$, while the circles O and O′ cut at an angle 2θ.

For § 3, I am indebted to an article in Mathesis ; but unfortunately I have lost the reference.

22nd August, 1911. W. GALLATLY.

The Conic of closest Contact.

The equation of the conic of closest contact at a given point of curve may be obtained as under :

Let P be the given point. Then the conic of closest contact, referred to the tangent and normal at P as axes, will be of the form

$$ax^2+2hxy+by^2=2y.$$

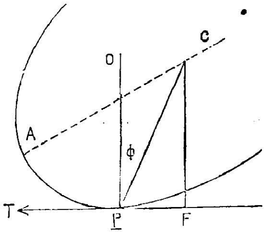
At the origin $\rho=\text{Lt.}\left(\frac{x^2}{2y}\right)$

$\therefore$ $a\rho=1$, or $a=\frac{1}{\rho}$ ... ... (i)

The centre of the conic lies on $ax+hy=0$; so that, if (x,y) be the centre

$\frac{x}{y}=-\frac{h}{a}$; but $\tan\phi=\frac{\text{PF}}{\text{CF}}=\frac{x}{y}$.

$\therefore$ $h=-a\tan\phi$.



Again, in a conic $\rho\propto\frac{1}{p^3}$.

Differentiating $\frac{1}{\rho}\cdot\frac{d\rho}{ds}=-\frac{3}{p}$

$$\frac{dp}{d\psi}\cdot\frac{d\psi}{ds}=-\frac{3}{p\rho}\frac{dp}{d\psi}.$$

$\therefore$ $\frac{d\rho}{ds}=-\frac{3}{p}\cdot\frac{dp}{d\psi}=3\tan\phi$ ... ... (ii)

since $x=\text{PF}=-\frac{dp}{d\psi}$, $y=\text{CF}=p$.

Thus $h=-a\tan\phi=-\frac{1}{3\rho}\frac{d\rho}{ds}$ ... ... (iii)

Differentiating (ii) with respect to ψ, we get

$$\frac{d^2\rho}{ds^2}\cdot\frac{ds}{d\psi}=\frac{3}{p^2}\left(\frac{dp}{d\psi}\right)^2-\frac{3}{p}\frac{d^2p}{d\psi^2}$$

$$=\frac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\frac{3}{p}\left(\rho-p\right), \text{ since } \rho=p+\frac{d^2p}{d\psi^2}.$$

$$=\frac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\frac{3\rho}{p}+3$$ ... ... (iv)

But $p=$ the ordinate of the centre of the conic,

$=\frac{a}{ab-h^2}$, by the usual formula.

$$\therefore \qquad b=\frac{1}{p}+\frac{h^2}{a}=\frac{2}{9\rho}\left(\frac{d\rho}{ds}\right)^2+\frac{1}{\rho}-\frac{1}{3}\frac{d^2\rho}{ds^2} \quad \ldots \quad \ldots \quad (v)$$

substituting for p, h, and a from (iv), (iii) and (i).

Hence, finally, the equation of the conic is

$$\frac{1}{\rho}x^2-\frac{2}{3\rho}\frac{d\rho}{ds}xy+\left\{\frac{1}{\rho}+\frac{2}{9\rho}\left(\frac{d\rho}{ds}\right)^2-\frac{1}{3}\frac{d^2\rho}{ds^2}\right\}y^2=2y.$$

1st April, 1912. D. D. KAPADIA.

Note on a set of Simultaneous Equations.*

1. Consider the equations

$$x_1+x_2+x_3+, \ldots\ldots\ldots\ldots + x_n = a_1,$$
$$x_1y_1+x_2y_2+x_3y_3+\ldots\ldots\ldots+x_ny_n=a_2,$$
$$x_1y_1^2+x_2y_2^2+x_3y_3^2+\ldots\ldots\ldots+x_ny_n^2=a_3,$$
$$x_1y_1^3+x_2y_2^3+x_3y_3^3+\ldots\ldots+x_ny_n^3=a_4,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_1y_1^{2n-1}+x_2y_2^{2n-1}+x_3y_3^{2n-1}+\ldots+x_ny_n^{2n-1}=a_{2n}$$

where $x_1, x_2, x_3, \ldots\ldots x_n$, and $y_1, y_2, y_3 \ldots\ldots y_n$ are $2n$ unknown quantities.

Now, let us take the expression

$$\phi(\theta)\equiv\frac{x_1}{1-\theta y_1}+\frac{x_2}{1-\theta y_2}+\frac{x_3}{1-\theta y_3}+\ldots\ldots+\frac{x_n}{1-\theta y_n} \quad \ldots \quad \ldots \quad (1)$$

and expand it in ascending powers of θ. Then we see that the expression is equal to

$$a_1+a_2\theta+a_3\theta^2+\ldots\ldots+a_{2n}\theta^{2n-1}+\&c. \quad \ldots \quad \ldots \quad (2)$$

But (1) when simplified will have for its numerator an expression of the $(n-1)^{th}$ degree in θ and for its denominator an expression of the nth degree in θ.

Thus we may suppose that

$$\phi(\theta)=\frac{A_1+A_2\theta+A_3\theta^2+\ldots\ldots+A_n\theta^{n-1}}{1+B_1\theta+B_2\theta^2+B_3\theta^3+\ldots\ldots+B_n\theta^n} \quad \ldots \quad \ldots \quad (3)$$
$$=a_1+a_2\theta+a_3\theta^2\ldots\ldots+a_{2n}\theta^{2n-1}+\&c.$$

$$\therefore \quad (1+B_1\theta+\ldots)(a_1+a_2\theta+\ldots)=A_1+A_2\theta+\ldots\ldots$$

Equating the coefficients of like powers of θ, we have,

$$A_1=a_1,$$
$$B_2=a_2+a_1B_1,$$
$$A_3=a_3+a_2B_1+a_1B_2,$$

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* For a solution, by determinants, of a similar set of equations, see Burnside and Panton: *Theory of Equations*, Vol. II, p. 106, Ex. 3. Ed.

$$A_n = a_n + a_{n-1}B_1 + a_{n-2}B_2 + \dots\dots + a_1B_{n-1},$$
$$0 = a_{n+1} + a_nB_1 + \dots\dots\dots + a_1B_n,$$
$$0 = a_{n+2} + a_{n+1}B_1 + \dots\dots + a_2B_n,$$
$$0 = a_{n+3} + a_{n+2}B_1 + \dots\dots + a_3B_n,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$0 = a_{2n} + a_{2n-1}B_1 + \dots\dots + a_nB_n.$$

From these $B_1, B_2, \dots B_n$ can be found easily, and since $A_1, A_2, \dots A_n$ depend upon these values they can also be found.

Now, splitting (3) into partial fractions in the form

$$\frac{p_1}{1-q_1\theta} + \frac{p_2}{1-q_2\theta} + \frac{p_3}{1-q_3\theta} + \dots\dots + \frac{p_n}{1-q_n\theta}$$

and comparing with (1), we see that

$$x_1 = p_1 \text{ and } y_1 = q_1$$
$$x_2 = p_2 \text{ and } y_2 = q_2$$
$$x_3 = p_3 \text{ and } y_3 = q_3$$
$$\cdot\ \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot$$
$$\&c., \qquad \&c.$$

2. As an example we may solve the equations :

$$x + y + z + u + v = 2$$
$$px + qy + rz + su + tv = 3$$
$$p^2x + q^2y + r^2z + s^2u + t^2v = 16$$
$$p^3x + q^3y + r^3z + s^3u + t^3v = 31$$
$$p^4x + q^4y + r^4z + s^4u + t^4v = 103$$
$$p^5x + q^5y + r^5z + s^5u + t^5v = 235$$
$$p^6x + q^6y + r^6z + s^6u + t^6v = 774$$
$$p^7x + q^7y + r^7z + s^7u + t^7v = 1769$$
$$p^8x + q^8y + r^8z + s^8u + t^8v = 5126$$
$$p^9x + q^9y + r^9z + s^9u + t^9v = 12595$$

where $x, y, z, u, v, p, q, r, s, t$ are the unknowns. Proceeding as before, we have

$$\frac{x}{1-\theta p} + \frac{y}{1-\theta q} + \frac{z}{1-\theta r} + \frac{u}{1-\theta s} + \frac{v}{1-\theta t}$$

$$= 2 + 3\theta + 16\theta^2 + 31\theta^3 + 103\theta^4 + 235\theta^5 + 774\theta^6 + 1769\theta^7 + 5126\theta^8 + 12595\theta^9 + \&c.$$

By the method of indeterminate coefficients, this can be shewn to be equal to

$$\frac{2 + \theta + 3\theta^2 + 2\theta^3 + \theta^4}{1 - \theta - 5\theta^2 + \theta^3 + 3\theta^4 - \theta^5}.$$

Splitting this into partial fractions, we get the values of the unknowns, as follows :—

$$x=-\frac{3}{5} \qquad p=-1$$

$$y=\frac{18+\sqrt{5}}{10} \qquad q=\frac{3+\sqrt{5}}{2}$$

$$z=\frac{18-\sqrt{5}}{10} \qquad r=\frac{3-\sqrt{5}}{2}$$

$$u=-\frac{8+\sqrt{5}}{2\sqrt{5}} \qquad s=\frac{\sqrt{5}-1}{2}$$

$$v=\frac{8-\sqrt{5}}{2\sqrt{5}} \qquad t=-\frac{\sqrt{5}+1}{2}$$

1st March, 1912. S. RAMANUJAN.

The Harmonic Centre.

1. Let $A_1, A_2, \ldots A_n$ be n points on a straight line, whose distances from an origin O on the same line, are $\rho_1, \rho_2 \ldots \rho_n$ determined by the equation

$$f(\rho) \equiv \rho^n - p_1 \rho^{n-1} + p_2 \rho^{n-2} - + \ldots (-1)^n p_n = 0. \quad \ldots \quad \ldots \quad (1)$$

Find a point B on the line such that the *sum* of the reciprocals of $BA_1, BA_2, \ldots BA_n$ is zero. That is, the distance OB is given by the equation

$$\frac{1}{\rho-\rho_1}+\frac{1}{\rho-\rho_2}+\ldots\ldots\frac{1}{\rho-\rho_n}=0, \quad \ldots \quad \ldots \quad (2)$$

which is an equation of the $(n-1)^{th}$ degree in ρ, shewing that there are $(n-1)$ points B.

These $(n-1)$ points B_r $[r=1, 2, \ldots (n-1)]$ may be called the *harmonic centres* of the n given points.

2. Equation (2) is obviously identical with $f'(\rho)=0$. Hence we have the following :—

The roots of $f'(\rho)=0$, determine the harmonic centres of the range given by $f(\rho)=0$.

The properties of $f''(\rho)=0$, $f'''(\rho)=0$, &c., are similarly derived. Thus, the roots of $f''(\rho)=0$, furnish $(n-2)$ points C_r, which may be styled *the hamonic centres of the second order* of the original set of points; and the sum of the products two together of the reciprocals of the distances of $A_1, A_2, \ldots A_n$ from any C is zero. For, by putting $\rho=\rho_c+c$, (where ρ_c stands for the distance of any A from C, and $c=OC$), it is seen

that the coefficient of ρ_c^2 in the resulting equation is zero. This may also be inferred from the property

$$\frac{f''(\rho)}{f(\rho)} = \underline{|2} \sum \frac{1}{(\rho-\rho_1)(\rho-\rho_2)}$$

3. Again, the *mean centre* of the given range is coincident with that of the systems of points defined by $f'(\rho)=0$, $f''(\rho)=0$, ..., since the arithmetic mean of the roots of any equation is the same as for $f(\rho)=0$. In other words, the several sets of harmonic centres have the same *mean centre*.

4. Let ρ denote *any plane vector* quantity. Then the equation $f(\rho)=0$, gives the vertices of a plane polygon, say A_r $(r=1, 2, \ldots n)$. In this case, the equations $f'(\rho)=0$, $f''(\rho)=0$, ... furnish definite sets of points *geometrically* related to the polygon, and possessing the following properties :—

(i) Each set has the same *mean centre*, which is identical with that of the vertices of the original polygon.

(ii) If $B_1 B_2 \ldots B_{n-1}$ be the polygon determined by $f'(\rho)=0$, the *vector* equation

$$\frac{1}{\overline{BA_1}} + \frac{1}{\overline{BA_2}} + \cdots \frac{1}{\overline{BA_n}} = 0,$$

is true for every B.

That is, forces acting at any B inversely proportional to its distances from the A's and along the reflexions of BA_r in any axis, are in equilibrium. Hence, also similar forces along BA_r $(r=1, 2, 3 \ldots n)$ are in equilibrium.

(iii) Similarly, for the C's determined by the equation $f''(\rho)=0$, and so on.

Finally, if S_1, S_2 correspond to $f^{n-2}(\rho)=0$, and M to $f^{n-1}(\rho)=0$, M is the common mean centre of the several polygons and is the middle point of $S_1 S_2$.*

5. Further properties of the vector equations

$$f'(\rho)=0, \; f''(\rho)=0, \ldots$$

lead to the theorems :

(i) The product $BA_1 \, BA_2 \ldots BA_n$ has astationary value for any B. Similarly, $CB_1 . CB_2 \ldots CB_{n-1}$ is stationary for any C ; and so on.

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* The beautiful theorem that '*if $f(\rho)$ is a cubic, then $f'(\rho)=0$, gives the foci of the maximum inscribed ellipse*' belongs to Professor Percy J. Heawood, M.A., and was kindly communicated to me by Mr. V. Ramaswami Iyer, M.A., sometime ago. The relation between the roots of $f(\rho)=0$ and $f'(\rho)=0$, was discussed by P. J. Heawood in the *Quarterly Journal of Mathematics*, 1905-6.

(ii) S_1, S_2 are the foci of the max. inscribed ellipse of the triangle given by $f^{n-3}(\rho)=0$; and the product of the distances of either S from the vertices of the triangle has a stationary value.

The *harmonic centres* of the vertices of a polygon $A_1 A_2 \ldots A_n$ may thus be defined as the points $B_r (r=1,2,\ldots n-1.)$ for which the product $BA_1 . BA_2 \ldots BA_n$ has a stationary value.

6. Let $r_1, r_2, \ldots r_n$ denote the distances of a point B from the vertices $A_1, A_2 \ldots A_n$, to show that the product $r_1 r_2 \ldots r_n$ is stationary when forces inversely proportional to $r_1, r_2, \ldots r_n$ acting at B are in equilibrium, we may proceed as follows:

The condition for a stationary value of the product is

$$\sum\left(\frac{\delta r_1}{r_1}\right)=0, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now, if B′ is a neighbouring position of B, denoting BB′ by δs, we have

$$\cos\phi_1=\frac{\delta r_1}{\delta s}, \ \&c.$$

∴ (1) becomes

$$\sum\left(\frac{\cos\phi_1}{r_1}\right)=0,$$

which means that the sum of the resolved parts of $\frac{1}{r_1}, \frac{1}{r_2} \ldots$ along BB′ is zero. Since BB′ may be in *any* direction, it follows that forces inversely proportional to $r_1, r_2 \ldots$ are in equilibrium.

7. More generally, if $A_1, A_2, \ldots A_n$ be points in space, their harmonic centres B_r $(r=1, 2, \ldots)$ may be defined by either of the properties:

(i) Forces, acting at any B inversely proportional to its distances from the A's are in equilibrium: what is the same thing, B is the mean centre of the inverses of the A's with respect to B.

(ii) The product of the distances of the A's from any B has a stationary value.

1st Feby., 1912. M. T. NARANIENGAR.

The Face of the Sky for July and August 1912.

Sidereal time at 8 p.m.

		July.				August.		
		H.	M.	S.		H.	M.	S.
1	... ...	14	37	9	...	16	39	22
8	... ...	15	4	45	...	17	6	58
15	... ...	15	32	21	...	17	34	34
22	... ...	15	59	57	...	18	2	10
29	... ...	16	27	33	...	18	29	46

From this table the constellations visible during the evenings of July and August can be ascertained by a reference to their positions as given in the star charts.

Phases of the Moon.

July.			H.	M.			August.		H.	M.	
Last Quarter	7	...	10	17	P.M.	...	6	...	9	48	A.M.
New Moon	14	...	6	43	„	...	13	...	1	28	„
First Quarter	21	...	10	48	A.M.	...	19	...	10	27	P.M.
Full Moon	22	...	9	58	„	...	28	...	1	29	A.M.

The Planets.

Mercury attains its greatest elongation (27° 5′ E) on July 25. It is stationary on August 7 and is in inferior conjunction with the Sun on August 22. It is again stationary on August 31. It is in conjunction with the Moon on July 16 and August 13.

Venus is in superior conjunction with the Sun on July 16 and is thereafter an evening Star. It is in conjunction with the Moon on July 14 at 9·2 8 P.M. and on August 13 at 6·1 P.M.

Mars is in conjunction with the Moon on July 17 and August 14 at 9·20 P.M.

Jupiter continues to retrogade. The planet is stationary on August 3, when it resumes direct motion and is in quadrature to the Sun on August 20 at 6·40 P.M.

Saturn is a morning star in July. It is in quadrature to the Sun on August 27. It is in conjunction with the Moon on July 11 and August 27.

1st *June*, 1912. V. RAMESAM.

REVIEW.

Introduction à la Théorie des Equations Intégrales: by Trajan Lalesco, Professor at Bucarest, Pp. 152, (Paris, A. Hermann et Fils, 1912).

The study of Integral Equations has not yet, as far as I am aware, been taken up by Indian mathematicians. The subject is, however, interesting, far reaching in its results, chiefly in Dynamics and Mathematical Physics, and opens a vast field for mathematical research. Indeed the efforts of the most eminent continental mathematicians have been for a great part, directed in this channel during the last fifteen years; and already many important results have been arrived at and have now become almost classic. These being, however, scattered over a numerous bibliography, the necessity for a didactic treatment of the whole subject has arisen, and it is gratifying to see that useful works of the kind begin to come forth. The reader will find in Mr. Lalesco's little book a fruitful mine of information and will thereby be initiated to this important branch of study.

Before giving a more detailed account of the book under review, we may remark that it is not a book for beginners; Mr. Lalesco writes for well informed mathematicians who are thoroughly acquainted with the recent advances and discoveries in mathematics. He supposes, therefore, that the reader is quitear with the famili whole range of Higher Mathematics and begins his subject abruptly.

The present volume is divided into three parts of greatly unequal length and value. The second part which is concerned with Fredholm's equation

$$\phi(x)+\lambda\int_a^b N(x,\ s)\phi(s)ds=f(x),$$

contains 82 pages and is by far the most important; and is rightly so. For, the fact that this equation may be used to solve the well-known problems of Dirichlet and many fundamental theorems of Physics in the theory of heat, electricity, etc., was undoubtedly the cause of the wonderful interest it created among mathematicians, and of the researches which were then started and still continue in functional equations.

In a very short introduction the author merely defines the meaning of Integral Equations and points out the different types which form the subject of his investigations. It may not be out of place to note here that in these equations of which Fredholm's equation is a type, the unknown function $\phi(s)$ is under the sign of integration; the other two functions are given, and $N(x, s)$, which is called the *kernel*, plays a most important part in the whole theory.

The first part which consists of a single chapter is short and gives a sketch of Volterra's works. The latter's theorem stating that, under certain conditions, $\int_0^x N(x,s)\,\phi(s)ds = f(x)$, has one and only one finite and continuous solution is elegantly treated by E. Picard's method of successive approximations. The notion of *resolving kernel*, which forms the basis of the solution of Integral Equations, is here introduced and the connection between Volterra's equations and Linear Differential Equations is also explained.

The second and fundamental part is divided into four very imports ant chapters and deals in a masterly way with Fredholm's equation mentioned above, in the case of functions which are finite and capable of integration.

Chapter I is concerned with the solution of Fredholm's first problem. The historical sketch of Fredholm's discovery and of the work of D. Hilbert and E. Schmidt in the study of Integral Equations will be read with interest. These two mathematicians treat the subject by using the theories of bilinear and quadratic forms which have been studied by Weierstrass, Sylvester, Kronecker and Frobenius, and by applying them on passing to infinity. Our author avoids this method and prefers to follow E. Goursat and B. Heywood. He thus obtains the two fundamental equations $D(\lambda)$ and $D(xy\lambda)$ of Fredholm, by forming integral equations of the *resolving kernel*, and expresses easily by their means the solution of the first problem. The study of the zeros of $D(\lambda)$ closes thi chapter. The *resolving kernel* is used throughout the first chapter.

Chapter II is devoted to a clear, elegant and thorough study of this kernel. Orthogonal and bi-orthogonal functions are at first introduced; and then a clever application of Goursat's orthogonal kernels leads to the general equation of resolving kernels. Goursat's properties of *principal functions* are also made use of; and when all these instruments have been well prepared, it is an easy task for the author to study directly a special kind of kernel which enables him to unfold the properties of the *fundamental functions* of a *canonical kernel*. The second and third theorems of Fredholm are then very easily derived. This chapter is certainly very important and the methods are simple and elegant.

Little need be said about the third chapter. It is concerned with the properties of various kinds of kernels. Symmetrical kernels hold a prominent place; their properties are clearly exhibited, and the author is able to introduce Bessel's inequalities and to prove Hilbert's beautiful

and fundermental theorems on the expansion in a series of *fundamental, functions.* Some other kinds of kernels are then considered.

Chapter IV deals with Fredholm's second equation

$$\int_a^b N(x,s)\,\phi(s)\,ds=f(x).$$

In the solution of this equation difficulties of altogether a different nature occur, and the author is obliged to abandon the notion of integral in the sense of Riemann, in order to adopt Lebesgue's integral. After defining *mean convergency* of a series of functions of real variables and studying some properties, he deduces the theorem of Fischer-Rietz, and extends to non-symmetrical kernels the important theorems on expansion in series. Picard's solution of the problem concludes this chapter.

Mr. Lalesco comes then to the third part of his work and collects in a single chapter the chief results of researches made in Integral Equations whose kernels have singularities. The study of such functions is not yet far advanced and the author is obliged to confine his work to the consideration of some well-chosen examples and to Picard's theorems. And he concludes the chapter by giving some indications on integral equations of a higher order and on the more recent researches of Volterra on *permutable functions.*

In an appendix the author has added a very complete list of the works treating of this subject.

Mr. Lalesco's treatise is certainly remarkable for its clearness and the beauty and simplicity of its methods. It contains in a didactic form the essential properties and the chief results of integral equations. With the help of this book, the reader will be able to study the numerous works which deal with the applications of the theory to Dynamics and Mathematical Physics.

1st *February,* 1912. C. PRUVOT.

SOLUTIONS.

Question 155.

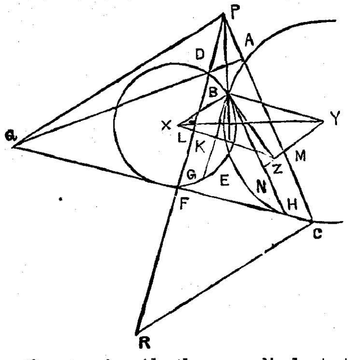
(T. RAJARAMA RAO, B.A., B.L.) :—(i) BE is the radical axis of the circles ABC and DEF. On EB produced take any point P and draw PAC parallel to the tangent drawn to the circle DEF at the point B, cutting the circle BAC at A,C. Similarly, draw PDF parallel to the tangent at B to the circle BAC, cutting the circle DEF at D and F. Join AD, CF and produce them to meet at Q. Show that PQ is parallel to the line drawn through B such that the intercepts made on the line by the two circles are equal.

Solution by N. Sanjiva Rao.

Let X and Y be the centres of the circles, and BG, BH the tangents to the same at B.

Through C draw CR parallel to PQ, and let CR meet PF produced at R.

Complete the parallelogram XBYZ, and draw its diagonals. Let XZ, YZ meet PF, BG ; BH, PC at L, K, N, M, respectively.



XY the line of centres is evidently perpendicular to the radical axis PBE. Since PDF is parallel to BG the tangent at B, BY the radius to the point of contact is also perpendicular to PF.

Therefore XZ which is parallel to BY is perpendicular to PF, and $B\hat{Y}X = B\hat{P}F = \beta$, say. Similarly, YZ is perpendicular to PC, and $B\hat{X}Y = B\hat{P}C = \alpha$, say.

If r_1 and r_2 be the radii of the circles, then $r_1 \sin \alpha = r_2 \sin \beta$. BG, BH are tangents to the circles at their point of intersection, therefore they divide—each the other circle—into similar segments.

$$\therefore \quad \frac{BG}{BH} = \frac{r_1}{r_2}, \text{ and } \frac{XK}{YN} = \frac{r_1}{r_2}.$$

Also, $LK = PB \sin \beta$, and $MN = PB \sin \alpha$,

$$\therefore \quad \frac{LK}{MN} = \frac{\sin \beta}{\sin \alpha} = \frac{r_1}{r_2}.$$

$$\therefore \quad \frac{XK - LK}{YN - MN} = \frac{XL}{YM} = \frac{r_1}{r_2}.$$

Hence, the chords at the distances XL and YM are also in the same ratio; *i.e.*, $\frac{DF}{AC} = \frac{r_1}{r_2}$ ... ... (1)

Again $PD \cdot PF = PB \cdot PE = PA \cdot PC$. $\therefore$ A, D, F, C are concyclic.

$$\text{Now} \quad \frac{PA}{PD} = \frac{PF}{PC}, \text{ and } \frac{QC}{QA} = \frac{QD}{QF}$$

$$\therefore \quad \frac{RA \cdot QC}{PD \cdot QA} = \frac{PF \cdot QD}{QF \cdot PC} = \frac{PR}{QC} \cdot \frac{QD}{PC}, \text{ because CR is parallel to PQ.}$$

$$\therefore \quad \frac{PR}{PC} = \frac{PA \cdot QC^2}{PD \cdot QA \cdot QD} = \frac{PA \cdot QC}{PD \cdot QF}, \text{ since } QA \cdot QD = QC \cdot QF.$$

Now, the line ADQ cuts the sides of the $\triangle$ PCF,

$$\therefore \quad QC \cdot PA \cdot DF = QF \cdot AC \cdot PD.$$

$$\therefore \quad \frac{QC \cdot PA}{PD \cdot QF} = \frac{AC}{DF} = \frac{r_2}{r_1}, \text{ by (1)}$$

$$\therefore \quad PR : PC = r_2 : r_1.$$

In the two $\triangle^s$ PRC and BYZ, the angles RPC, BYZ are equal and the sides about these angles are proportional. The triangles are therefore similar.

But BY, YZ are respectively perpendicular to PR, PC. Therefore BZ is perpendicular to RC; *i.e.*, to PQ which is parallel to RC.

Since BZ is the diagonal of the parallelogram, it must pass through the middle point of XY and it is readily seen that the line perpendicular to BZ through B is such that the circles make equal intercepts on this line.

Hence PQ is parallel to this line.

Question 201.

(M. Bhimasena Rao.) :—P is a common point of the director circles of the system of conics inscribed in a quadrilateral whose diagonal triangle is ABC. Shew that the pedal circle of P with respect to the triangle ABC touches the pedal circles with respect to the same triangle

of isogonal conjugate points on lines through P perpendicular to the sides of the quadrilateral. Apply this theorem, when P is (1) the orthocentre, (2) the circum-centre of the triangle ABC.

Remarks by the Proposer.

A simple proof of this question is given by Mr. S. Purushottam in the Journal for February 1911. The following analysis gives the points of contact of the circles :

Let A′B′C′ be the pedal triangle of P and S, S′ isogonal conjugate points on any line through P making angles λ, μ, ν, with the sides of ABC. Let X, X′ be the projections of S, S′ on BC and t_1, t_2, t_3, denote the tangents from A′, B′, C′, to the pedal circle of S. Then $t_1=\sqrt{A'X\cdot A'X'}=\cos\lambda\sqrt{PS\cdot PS'}$. Similarly $t_2=\cos\mu\sqrt{PS\cdot PS'}$, and $t_3=\cos\nu\sqrt{PS\cdot PS'}$. The pedal circle of S will touch the pedal circle of P only when

$$a't_1\pm b't_2\pm c't_3=0,$$

where a',b',c', denote the sides of A′B′C′ ; *i.e.*, when

$$a'\cos\lambda\pm b'\cos\mu\pm c'\cos\nu=0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now PS passes through ($\sin\lambda$, $\sin\mu$, $\sin\nu$) at infinity. When (1) is satisfied, one of the lines $a'\alpha\pm b'\beta+c'\gamma=0$, passes through ($\cos\lambda$, $\cos\mu$, $\cos\nu$). In other words, PS is perpendicular to one of the lines $a'\alpha\pm b'\beta\pm c'\gamma=0$, which may be drawn thus: Bisect the angles BPC, CPA, APB internally and externally, the bisectors meeting BC, CA, AB in L, L′ ; M, M′ ; N, N′. The lines L′M′N′, MNL′, NLM′ LMN′ are represented by

$$a'\alpha\pm b\beta\pm c'\gamma=0.$$

The points of contact: let PS be perpendicular to L′M′N′, *viz.*, $a'\alpha+b'\beta+c'\gamma=0$, and F the point of contact. Then

$$FA' : FB' : FC'=t_1 : t_3 : t_3=\cos\lambda : \cos\mu : \cos\nu$$
$$=bc'-b'c : ca'-c'a : ab'.-a'b$$

Referred to A′B′C′, the normal coordinates of F are

$$\left(\frac{1}{b'c-bc'}, -\frac{1}{c'a-ca'}, \frac{1}{a'b-ab'}\right) ; \text{ etc.,}$$

(i) Equation to FF_1 is

$$a'\alpha\,(b'c-bc')^2-b'\beta\,(c'^2a^2-c^2a'^2)+c\gamma\,(a'^2b^2-a^2b'^2)=0.$$

Equation to F_2F_3 is

$$-a'\alpha\,(b'c+bc')^2+b'\beta\,(c'^2a^2-c^2a'^2)-c'\gamma\,(a'^2b^2-a^2b'^2)=0.$$

Hence FF_1 and F_2F_3 intersect at G ; *viz.*,

$$\left\{0, \frac{1}{b'\,(c^2a'^2-c^2a'^2)}, \frac{1}{c'\,(a'^2b^2-a^2b'^2)}\right\}$$

on B′C′.

Similarly H, K the remaining diagonal points of the quadrangle $FF_1F_2F_3$ lie on A′C′ and A′B′.

(ii) A′G, B′H, C′K are concurrent at T, *viz.*,

$$\left\{\frac{1}{a'(b'^2c^2-b^2c'^2)}, \frac{1}{b'(c'^2a^2-c^2a'^2)}, \frac{1}{c'(a'^2b^2-a^2b'^2)}\right\}$$

(iii) T lies on the circle A′B′C′; for $\Sigma a'^2(b'^2c^2-b^2c'^2)=0$.

(iv) T lies also on the nine-point circle of ABC, and is the orthopole of the diameter through P. This cannot be proved here.

When P is the ortho-centre or circumcentre of ABC, we easily get Feuerbach's theorem and its extension by M'Cay. In the case of the Feurbach points, (i) is given by Mr. M. T. Naraniengar in his *Note on the Nine-point circle* (Vol. I, J.I.M.S.)

Question 283.

(S. Ramanujan) :—Shew that it is possible to solve the equations:

$$x+y+z=a, \qquad p^3x+q^3y+r^3z=d,$$
$$px+qy+rz=b, \qquad p^4x+q^4y+r^4z=e,$$
$$p^2x+q^2y+r^2z=c, \qquad p^5x+q^5y+r^5z=f,$$

Where x, y, z, p, q, r are the unknowns. Solve the above when $a=2, b=3, c=4, d=6, e=12, f=32$.

[For the solution of a more general question, see Note by the Proposer on p. 94 above.]

Question 296.

(J. C. Swaminarayan, M.A.):—If $x^2+y^2+z^2=x'^2+y'^2+z'^2=r^2$ find the limiting value of

$$\frac{\lambda(yz'-y'z)+\mu(zx'-z'x)+\nu(xy'-x'y)}{\sqrt{\{(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2\}}}$$

when $x=x', y=y', z=z'$.

Remarks by H. Br.

The problem is defective.

If O be the origin, P,P′ the points (x,y,z), (x',y',z') respectively, and A the point (λ,μ,ν), the given expression is equal to the length of the perpendicular from A on the *plane* OPP′. It is therefore independent of the distance PP′. The given condition (OP = OP′ = r) is also irrelevant.

Question 309.

(M. BHIMASENA RAO):—Shew that the Feuerbach's tangents also touch the ellipse which is the locus of the orthopoles of lines through the ortho-centre of the fundamental triangle.

Solution by the Proposer, and S. Narayanan, B.A., L.T.

The trilinear equation of the locus of orthopoles of lines passing through any point, when referred to the pedal triangle of that point, has been shown to be $\frac{a^2}{a'\alpha}+\frac{b^2}{b^2\beta}+\frac{c^2}{c'\gamma}=0$, where a', b', c' denote the sides of the pedal triangle, see solution of question 238, p. above. If the ABC-*normal c's* of any point be $(l, m, n,)$, its *normal c's* referred to the ortho-centric triangle are

$-l\cos A+m\cos B+n\cos C$, $l\cos A-m\cos B+n\cos C$, $l\cos A+m\cos B-n\cos C$.

The in-Feuerbach point is the centre of the rect hyp. self conjugate with respect to the ortho-centric triangle and passing thro' $(-\cos A+\cos B+\cos C$, etc.).

This rect. hyp. is

$$\Sigma\alpha^2\{(\cos A-\cos B+\cos C)^2-(\cos A+\cos B-\cos C)^2\}=0$$

$$i.e\ \Sigma\alpha^2\cos A\,(\cos B-\cos C)=0.$$

Its centre is

$$\frac{a'}{\cos A(\cos B-\cos C)},\ \frac{b'}{\cos B(\cos C-\cos A)},\ \frac{c'}{\cos C(\cos A-\cos B)}$$

where a', b', c' are the sides of the orthocentric Δ.

The nine-point circle is $\frac{a'}{\alpha}+\frac{b'}{\beta}+\frac{c'}{\gamma}=0.$

The F-tangent is $\sum\frac{\alpha}{a}\cos^2 A(\cos B-\cos C)^2=0,$

$$i.e.,\quad \sum\frac{a^2\alpha}{a'}.\left\{\frac{\cos A(\cos B-\cos C}{a}\right\}^2=0.$$

This shows that the F-tangent touches the locus $\sum\frac{a^2}{a'\alpha}=0$, at

$\frac{a}{\cos A(\cos B-\cos.C)}$, etc., which may be shown to be the orthopole of the line joining the ortho-centre and the incentre.

Similarly, for the remaining F-tangents.

Question 313.

(A. Krishnaswami Ayengar) :—TQ, TQ′ are tangents to the ellipse whose centre is C and foci are S,S′. If the circle TQQ′ touches the ellipse at P, prove that

(1) TP is half the central chord of curvature at P,
(2) the centre of curvature at P lies on the circle TQQ′,
(3) T lies on the circle SPS′,
(4) $CP.CT = CS^2$.

Solution by N. B. Pendse and the Proposer.

The polar of T for the ellipse passes thro X.

∴ The polar of X passes thro' T.

∴ PT is the polar of X for the ellipse.

Also XP is the tangent to the ⊙; and if PT cut QQ′ in Y, QYQ′X form a harmonic range.

∴ PT is the polar of X for the circle.

∴ XT must be the tangent at T to the circle.

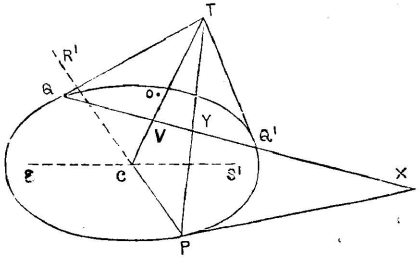
Because TQPQ′ is cyclic, QQ′ and the tangent at P are equally inclined to the axis.

∴ CP, CT the diameters conjugate to these are equally inclined to CS.

If CT meet QQ′ in V, V is the middle point of QQ′. Therefore if O is the circum-centre of TQQ′, OVTXP is cyclic.

Now $\quad OVX = OTX = OPX = \frac{\pi}{2}$.

∴ $\quad X\hat{P}T = X\hat{V}T$.



Also, because XP, XQ′ are equally inclined to the axis as well as $\overline{CP, CT}$.

$$T\hat{V}X = \text{the ext. angle of } CPX.$$

∴ CP, PT are equally inclined to tangent the at P, and therefore to S′P, SP.

In the Δ SPS′, because C is the middle point of SS′, and CP,PT are equally inclined to SP, S′P and CP,CT are equally inclined to SS′ the four points S,P,S′,T. are cyclic (a simple property of a circle)

$$\therefore S'\hat{P}T = C\hat{S}T.$$

But $S'\hat{P}T = CPS$. $\therefore C\hat{S}T = C\hat{P}S$. Also $S\hat{C}T = S\hat{C}P$.

$\therefore \Delta^s$ SCT, PCS are similar, so that

$$CT : CS = CS : CP, \text{ or } CP \cdot CT = CS^2.$$

Again Δ^s S′PC, SPT are similar

$$\therefore \quad S'P : CP = PT : SP$$

$$\therefore \quad PT = \frac{SP \cdot S'P}{CP} = \frac{CD^2}{CP}.$$

Thus PT = half the central chd. of curvature thro' P.

If PC meets the ⊙ TQQ′ in R′, then PR′ being the reflection of PT in the diam. of the circle thro' P, is equal to it.

Also, the diam. of the ⊙ thro' P is along the normal to the ellipse at P.

∴ The diam. of the ⊙ of curvature, and the central chord. of curvature lie along the normal at P and PR′ respectively.

But PR′ = PP′ = half the central chord of curvature.

∴ The diam. of the circle = half the diam of curvature.

Hence, the centre of curvature at P lies on the circle TQQ′.

Analytical Solution by N. *Sankara Aiyar.*

Question 238.

(M. BHIMASENA RAO.) —Find the locus of orthopoles, with respect to a triangle, of lines through a given point. Determine the position of the given point when the locus is (1) a circle, and (2) an inconic of the triangle.

Question 321.

(S. NARAYANAN, B.A., L.T.) :—Show that the join of the orthopoles of any two perpendicular straight lines bisects and is bisected by the join of their point of intersection with the orthocentre of the triangle.

Solution by M. Bhimasena Rao.

Let A′B′C′ be the pedal triangle of the given point P and λ, μ, ν be the angles made by any line through P, with the sides of the given triangle ABC. The orthopole of the line is the double point of A′B′C′ and A″B″C″—the pedal triangle of any other point on the line, and is therefore the mean centre of A′,B′,C′ for the multiples

$$\frac{\sin A}{A'A''}, \frac{\sin B}{B'B''}, \frac{\sin C}{C'C''}$$

i.e., for the multiples $a \sec \lambda$, $b \sec \mu$, $c \sec \nu$, see my 'Note on Double Points' in the Journal for February 1912.

The A′B′C′—*areal c's* of the orthopole are therefore

$$a \sec \lambda,\ b \sec \mu,\ c \sec \nu.$$

Since $a \cos \lambda + b \cos \mu + c \cos \nu = 0$, the areal equation of the locus is

$$\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The trilinear equation is $\dfrac{a^2}{a'\alpha}+\dfrac{b^2}{b'\beta}+\dfrac{c^2}{c'\gamma}=0.$

The circumcircle of A′B′C′ is $\dfrac{a'^2}{\alpha}+\dfrac{b'^2}{\beta}+\dfrac{c'^2}{\gamma}=0$, and (1) is a circle when $a' : b' \cdot c' = a : b : c$, *i.e.*, when P is the circumcentre of ABC.

The orthopole of the line through P perpendicular to the first line is (a cosec λ, b cosec μ, c cosec ν.)

The line joining the two orthopoles is

$$\alpha \sin \lambda \cos \lambda + \beta \sin \mu \cos \mu + \gamma \sin \nu \cos \nu = 0.$$

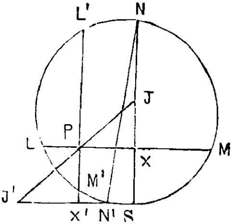
This passes through $a \cos A$, $b \cos B$, $c \cos C$, which is the centre of (1).

If the ABC-*normal c's* of P be l, m, n. Then those of the centre of the locus are

$$b \cos B\,(l + m \cos C) + c \cos C\,(l + n \cos B)\,;\ \text{etc.}$$

i.e., $b \cos B \cos C\,(al+bm+cn) + l\,(b \cos B + c \cos C - a \cos B \cos C)$, etc.
i.e., $\cos B \cdot \cos C\,(al+bm+cn) + l.\ 2R \sin A \sin B \sin C$; etc.

Hence, the centre is the mid-point of the line joining the ortho-centre of ABC and P.



Geometrical Proof: Let LM and L′ M′ be perpendicular chords thro' any point P; and N, N′ be the isogonal conjugates of the points at infinity on LM and L′M′. Let J be the ortho-centre of LMN and let NJ meet LM in X, and the circumcircle in S.

Since LM and L′ M′ are at right angles, the points N, N′ are diametrically opposite. Therefore SN′ is parallel to LM and perpendicular to L′ M′.

If SN′ meets L′ M′ in X′ and we take J′ on SN′ such that J′ X equals SX′, J′ is the ortho-centre of L′ M′ N′.

Now PXSX′ is a $\square^m$ and J X=XS, J′ X′=X′ S. Therefore JJ′ is bisected at P.

If H be the ortho-centre of the fundamental triangle; the ortho′ poles ω and ω' are the mid points of HJ and HJ′, so that H ω P ω' is a parallelogram.

Hence, HP and $\omega\omega'$ bisect each other.

Question 324.

(R. Srinivasan, M.A.) :—Find the value of

$$\sum_{m=1}^{\infty} \frac{(-1)^m \cos mx}{(m+1)(m+2)}$$

Solution by V. B. Naik, S. Ramanujan and G. Ramachandran.

Writing $$P=\sum_{m=1}^{\infty} \frac{(-1)^m \cos mx}{(m+1)(m+2)},$$

and $$Q=\sum_{m=1}^{\infty} \frac{(-1)^m \sin mx}{(m+1)(m+2)};$$

we have $$P+iQ=\sum_{m=1}^{\infty} \frac{(-1)^m e^{imx}}{(m+1)(m+2)}.$$

Now, the series

$$\sum_{m=1}^{\infty} \frac{(-1)^m z^m}{(m+1)(m+2)} = -\frac{z}{2\cdot 3}+\frac{z^2}{3\cdot 4}-\frac{z^3}{4\cdot 5}+ \quad \ldots$$

$$= -z\left(\frac{1}{2}-\frac{1}{3}\right)+z^2\left(\frac{1}{3}-\frac{1}{4}\right)-z^3\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots$$

$$= \left(-\frac{z}{2}+\frac{z^2}{3}-\frac{z^3}{4}+\ldots\right)+\left(\frac{z}{3}-\frac{z^2}{4}+\frac{z^3}{5}-\ldots\right)$$

$$= \frac{\log(1+z)}{z}-1+\frac{\log(1+z)}{z^2}-\frac{1}{z}+\frac{1}{2},$$

$$= \log(1+z)\left\{\frac{1}{z}+\frac{1}{z^2}\right\}-\frac{1}{z}-\frac{1}{2}.$$

Putting $z=e^{ix}$, we have

$$P+iQ=\log(1+e^{ix})\{e^{-ix}+e^{-2ix}\}-e^{-ix}-\tfrac{1}{2},$$

$$=\left[\log\left((2\cos\tfrac{x}{2}\right)+i\,\tfrac{x}{2}\right]\left[\cos x+\cos 2x-i(\sin x+\sin 2x)\right]$$
$$-\cos x+i\sin x-\tfrac{1}{2},$$

Equating the real parts,

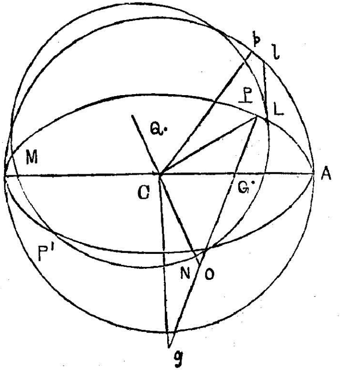
$$P=\log\left(2\cos\tfrac{x}{2}\right)\left[\cos x+\cos 2x\right]+\tfrac{x}{2}\left[\sin x+\sin 2x\right]-\cos x-\tfrac{1}{2}.$$

Question 331.

(V. Ramaswami Aiyar, M. A.) :—Give a construction for the trisection of an angle by means of a *given* ellipse.

Solution by N. Durairajan.

Let P be the point on the ellipse whose eccentric angle is equal to the given angle ϕ. P′ is the other end of the diameter through P and O is the mid-point of Gg, with the usual notation.



The eccentric angles of the feet of the other three normals, OL, OM ON are $\frac{\pi-\phi}{3}$, $\frac{3\pi-\phi}{3}$, $\frac{5\pi-\phi}{3}$ (*vide* : J. I. M. S., Vol. II, p. 204, Q. 213.)

The difference of each of these is $2\pi/3$. Therefore the three feet form a maximum $\triangle$ in which the tangent at any point is parallel to the opp. side. Hence O is the orthocentre of $\triangle$ LMN, and C is known to be the centroid. It is also known that LMNP′ is a cyclic quadrilateral,

Hence we have the following construction :—Produce OC to Q, such that $CQ = \frac{1}{2}OQ$. Describe a circle, with centre Q and radius QP′, cutting the ellipse in L, M, N. From the eccentric angle of L, &c.' we get $\phi/3$.

Question 332.

(V. Ramaswami Aiyar, M A. :—Four conics are circumscribed to a triangle having for directrices the lines joining the orthocentre to the in-and ex-centres of the triangle. Show that the foci of the conics corresponding to these directrices, lie on the circumcircle and are the inverses of the Feuerbach points with respect to the polar circle of the triangle.

Solution by M. Bhimasena Rao.

Let H, O, I be the ortho-centre, circum-centre and in-centre. Also let S be the centre of the self conjugate rectangular hyperbola touching HI. The polar reciprocal of the incircle with respect to this rect. hyp. is a circum conic whose focus is S_2 and whose directrix is the polar of I with respect to the rect. hyp., *viz.*, HI. The polar of H with respect to the rect. hyp. is OI, since the polar of any point with respect to a self-conjugate rect. hyp. passes through its isogonal conjugate. The polar reciprocals of the inconics touching OI are circumconics passing through H, and therefore rectangular hyperbolas. Hence, S is a common point of the director circles of the inconics touching OI.

Now, it is known that the director circles of these inconics pass through the internal Feuerbach point and its inverse with respect to the polar circle, and that the director circle of the inconic which passes through O, touches the nine point circle and the circumcircle at these points.

Hence, S which lies on the circumcircle (for the centre of any self-conjugate rect. hyp. is on the circumcircle) is the inverse of the Feuerbach point.

Similarly, for the excentres.

Question 334.

(M. T. Naraniegar, M.A.):—If (h, k) be the mean centre of the points of intersection of the second polar line of any point with respect to $x^3 + y^3 = a^3$, and (h', k') the middle point of the chord of contact of the same point with respect to the first polar conic, prove that:

$$h/h' + k/k' = 1.$$

Solution by D. Jagadisan, M. Satyanarayana and others.

Let (l, m) be the point of which the first polar conic and the second polar line with respect to the cubic $x^3+y^3=a^3$, are taken.

The equation of the polar conic is

$$lx^2+my^2=a^3 \qquad \dots \qquad \dots \qquad (1)$$

The equation of the polar line which is also the chord of contact of tangents from (l, m) to the conic (1) is

$$l^2x+m^2y=a^3 \qquad \dots \qquad \dots \qquad (2)$$

The abscissae of the intersections of the polar line (2) and the cubic are given by the cubic equation

$$m^6x^3+(a^3-l^2x)^3=m^6a^3.$$

$$\therefore \qquad h=\frac{x_1+x_2+x_3}{3}=\frac{a^3l^4}{l^6-m^6}. \text{ Similarly } k=\frac{a^3m^4}{m^6-l^6}$$

The abscissae of the intersections of the chord of contact (2) and the conic (1), are given by the quadratic

$$lm^3x^2+(a^3-l^2x)^2=m^3a^3$$

$$\therefore \qquad h'=\frac{x_1+x_2}{2}=\frac{a^3l}{l^3+m^3}; \text{ similarly } k'=\frac{a^3m}{l^3+m^3}$$

$$\therefore \qquad \frac{h}{h'}+\frac{k}{k'}=\frac{l^3}{l^3+m^3}+\frac{m^3}{l^3+m^3}=1$$

Question 335.

(K. J. SANJANA, M.A.) :—Prove the following identity :

$$4a^2p^2-(b^2+q^2-c^2-r^2)^2+4b^2q^2-(c^2+r^2-a^2-p^2)^2+$$
$$4c^2r^2-(a^2+p^2-b^2-q^2)^2=\Pi_{abc}+\Pi_{aqr}+\Pi_{brp}+\Pi_{cpq},$$

where $\Pi_{xyz}\equiv(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$; and point out its relation to the quadrilateral ABCD for which $BC=a$, $CA=b$, $AB=c$, $DA=p$, $DB=q$, $DC=r$.

Solution by K. D. Shah, J. C. Swaminarayan and N. B. Pendse.

Simplifying the expression on the left hand side, we have

$$(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)+(2q^2r^2+2a^2q^2$$
$$+2a^2r^2-a^4-q^4-r^4)+etc.$$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$
$$+(a+q+r)(q+r-a)(r+a-q)(a+q-r)$$
$$+(b+r+p)(r+p-b)(p+b-r)(b+r-p)$$
$$+(c+p+q)(p+q-c)(q+c-p)(c+p-q)$$
$$=\pi_{abc}+\pi_{aqr}+\pi_{brp}+\pi_{cpq}$$
$$=16(\Delta^2+\Delta_1^2-\Delta_2^2+\Delta_3^2),$$

where Δ, Δ_1, Δ_2, Δ_3 denote the areas of the triangles ABC, BCD, CDA, DAB respectively.

If S_1, S_2, S_3 denote the areas of the three quadrilaterals that can be formed by taking different permutations of ABCD, the left hand member may also be written $16(S_1^2+S_2^2+S_3^2)$.

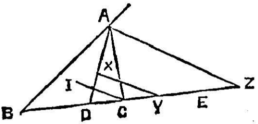
Question 337.

S. NARAYANAN, B.A., L.T. :—ABC is a triangle. D and E are the points of contact with BC of the incircle and the excircle corresponding to B, and I is the incentre. If X, Y be the midpoints of AD, CE respectively, shew that IC is parallel to XY.

Additional Solution by N. B. Pendse.

Find a point Z such that DZ=2DY.

Then $BD=s-b$, $DC=s-c$, $CE=s-a$, $\therefore CY=\frac{s-a}{2}$



Now, $DC+DE=2DY=DZ$

$\therefore$ DE=CZ

But $DE=b=CA$

$\therefore$ CZ=CA

$\therefore$ AZ is parallel to CI

$\therefore$ XY is also parallel to CI.

Question 343.

(M. KANNAN, B.A., L.T.) :—If ρ_1, ρ_2, ρ_3 be the radii of curvature at the points of contact of parallel tangents to a cardioid, shew that $\rho_1 \pm \rho_2 \pm \rho_3 = 0$.

Solution by R. Srinivasan, N. B. Pendse, and others.

Let the cardioid be $r=a(1-\cos\theta)$. Then the slope of the tangent at any point θ is readily seen to be $3\theta/2$. Therefore the tangents at the points $\theta, \theta+\frac{2\pi}{3}, \theta+\frac{4\pi}{3}$ are all parallel. Now $\rho=4a/3.\sin\theta/2$ at the point θ.

$$\therefore \rho_1-\rho_2+\rho_3=\frac{4a}{3}\left\{\sin\theta/2-\sin\left(\frac{\theta}{2}+\frac{\pi}{3}\right)+\sin\left(\frac{\theta}{2}+\frac{2\pi}{3}\right)\right\}=0.$$

Remarks by M. Satyanarayana, M. A.

The following question from the Ed. Times, is more general :

" If a system of parallel tangents be drawn to a plane curve, the sum of the radii of curvature at the points of contact is zero." [Q. 12313 by Mr. J. H. Grace.]

Question 344.

(V. V. Satyanarayana) :—The incircle of a triangle ABC touches BC at D and X is the middle point of BC. Shew that AD is bisected by XI.

Solution (1) by K. J. Sanjana, M.A., and K. Subramoney Aiyar ;
(2) by S. Naragan, B.A., L.T., and N Sanjiva Rao.

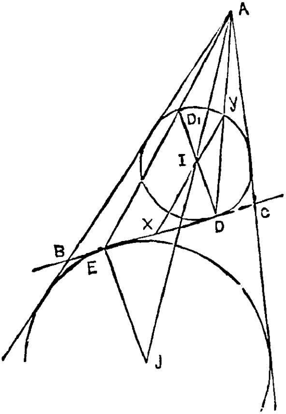
(1) Let E be the point of contact with BC of the first excircle, J its centre. Then JE, ID_1 are parallel radii, and ED_1 goes through A. the external centre of similitude. Also $BE=CD=s-c$, and $BX=XC$, $\therefore EX=XD$; and $DI=ID_1$, so that XI is parallel to EA and hence bisects DA.

(2) Let Y lie the middle point of AD.

We have $AB-BD=AC-CD$.

$\therefore \quad \triangle IAB-\triangle IBD=\triangle IAC-\triangle ICD$

$\therefore \quad 2.\ \triangle IBY=2.\ \triangle ICY$



Hence in the $\triangle IBC$, IY passes through the middle point of BC.

Other Solutions by J. C. Swaminarayan M.A., N. Sankara Rao, and G. Ramachandran.

Question 347.

(Professor Sanjana, M.A.) :—Denoting by s_r the sum of the series $\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}$...ad inf. ($r$ being a positive integer), prove that

$$\int_0^{\pi} \theta^k \log \operatorname{cosec} \theta .\, d\theta = \frac{\pi^{k+1}}{k+1} \log 2 + \frac{k}{2^2}\pi^{k-1}s_3 - \frac{k(k-1)(k-2)}{2^4}\pi^{k-3}s_5 + \frac{k(k-1)(k-2)(k-3)(k-4)}{2^6}\pi^{k-5}s_7 - \dots$$

the series to stop short of the term in which the index of π, or the co-efficient is zero.

Solution by J. C. Swaminarayan, M.A., and N. Sankara Aiyar.

$$\int_0^{\pi} \theta^k \cos 2n\theta d\theta = \left[\frac{\theta^k \sin 2n\theta}{2n}\right]_0^{\pi} - \frac{k}{2n}\int_0^{\pi} \theta^{k-1} \sin 2n\theta d\theta$$

$$= \left[\frac{k}{2^2 n^2}\theta^{k-1}\cos 2n\theta\right]_0^{\pi} - \frac{k(k-1)}{2^2 n^2}\int_0^{\pi} \theta^{k-1}\cos 2n\theta d\theta$$

Hence, if I_k denotes the integral $\int_0^{\pi} \theta^k \cos 2n\theta d\theta$, we have

$$I_k = \frac{k}{2^2 n^2}\pi^{k-1} - \frac{k(k-1)}{2^2 n^2} I_{k-2}$$

$$= \frac{k}{2^2 n^2}\pi^{k-1} - \frac{k(k-1)(k-2)}{2^4 n^4}\pi^{k-3} + \frac{k(k-1)(k-2)(k-3)(k-4)}{2^6 n^6}\pi^{k-5} - \dots$$

$$\therefore \int_0^{\pi} \theta^k \log \operatorname{cosec} \theta \, d\theta = \int_0^{\pi} \theta^k \left[\log 2 + \sum_1^{\infty} \frac{\cos 2n\theta}{n}\right] d\theta$$

$$= \frac{\pi^{k+1}}{k+1}\log 2 - \sum_1^{\infty} \frac{I_k}{n}$$

$$= \frac{\pi^{k+1}}{k+1}\log 2 + \sum_1^{\infty} \left[\frac{k}{2^2 n^3}\pi^{k-1} - \frac{k(k-1)(k-2)}{2^4 n^5}\pi^{k-3} + \frac{k(k-1)(k-2)(k-3)(k-4)}{2^6 n^7}\pi^{k-5} - \dots\right]$$

$$= \frac{\pi^{k+1}}{k+1}\log 2 + \frac{k}{2^2}\pi^{k-1}s_3 - \frac{k(k-1)(k-2)}{2^4}\pi^{k-3}s_5 + \frac{k(k-1)(k-2)(k-3)(k-4)}{2^6}\pi^{k-5}s_7 - \dots$$

QUESTIONS FOR SOLUTION.

374 (M. BHIMASENA RAO):—ABC and A′B′C′ are conjugate triangles with respect to a conic whose centre is S. If P is the centre of perspective of the triangles, show that

(1) The conics ABCPS and A′B′C′PS intersect in points D_1 and D_2 on the line at infinity;

(2) D_1 and D_2 are in the directions of conjugate diameters of the given conic;

(3) If Q be any point on ABCPS, lines through A′, B′, C′ parallel to AQ, BQ, CQ respectively are concurrent on A′B′C′PS.

(4) If QP intersects A′B′C′PS at Q′, the points Q and Q′ are corresponding points of the given triangles.

•**375**. (KESAVALAL, D. SHAH):—Find the rational integral factors of x^8-4x^4+16, shewing how the method adopted is suggested.

376. (ZERO):—In the limacon $r=a+b\cos\theta$, if r_1, r_2, r_3, r_4 donote the radii to the points of contact of parallel tangents, prove that

(i) $r_1+r_2+r_3+r_4=3a$,

(ii) $\Sigma(r_1 r_2)$ is invariable.

377. (J. C. SWAMINARAYAN, M.A.):—Shew that the value of the determinant—

$$\begin{vmatrix} a, & h, & g, & l, & l' \\ h, & b, & f, & m, & m' \\ g, & f, & c, & n, & n' \\ l, & m, & n, & o, & o \\ l', & m', & n', & o, & o \end{vmatrix}$$

can be expressed in either of the forms

(1) $\Sigma a(mn'-m'n)^2+2\Sigma f(nl'-n'l)(lm'-l'm)$,

(2) $\frac{1}{\Delta}\Big[(\Sigma Al^2+2\Sigma Fmn)(\Sigma Al'^2+2\Sigma Fm'n')-$

$$\{\Sigma A\,ll'+\Sigma F(mn'+m'n)\}^2\Big],$$

where corresponding capital letters denote the minors, and Δ denotes the value, of the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

378. (J. C. SWAMINARAYAN, M. A.) :—Having given that

$$\{ \Sigma(b+c)\, x^2 - 2\Sigma fyz) \}^2.$$

$$= 4\,(x^2+y^2+z^2)\,[(\Sigma bc-f^2)\, x^2+2\, \Sigma(gh-af)\, yz],$$

prove that $\dfrac{bz^2+cy^2-2\,fyz}{y^2+z^2} = \dfrac{cx^2+az^2-2\,gzx}{z^2+a^2} = \dfrac{ay^2+bx^2-2\,hxy}{x^2+y^2}$.

379. (A. C. L. Wilkinson, M. A.) :—Show that the property of a hyperbola that the portion of the tan ent at any point intercepted between the asymptotes is bisected at the point of contact and makes a triangle of constant area with the asymptotes, can be generalized for three dimensions, as follows :—

> Any triangle such that its sides touch a hyperboloid of two sheets and its vertices lie on the asymptotic cone is such that (1) the points of contact with the hyperboloid are the middle points of the sides, (2) the volume of the tetrahedron formed by the triangle and centre of the hyperloloid is constant.

Show also that if PQR is such a triangle and O is the centre of the hyperboloid, then $OP^2+OQ^2+OR^2-PQ^2-PR^2-QR^2$ is the algebraic sum of the squares of the axes of the hyperboloid.

380. (A. C. L. WILKINSON, M. A.) :—At the extremity A of the major (or minor) axis of a conic S, the circle of curvature C is drawn. PQ is any chord of the conic S which touches C, and T is its pole with respect to the conic S. A conic S′ is described through PQT touching the conic S at A. Show that the tangents at P,Q to S′ also touch the circle C, and that the radius of curvature of the conic S′ at A is half the radius of the circle C.

381. (V. RAMASWAMI AIYAR, M.A.) :—O is the circumcentre of a triangle ABC, PQ is any diameter of an inscribed conic. If D be a point in BC such that the angles AOD and POQ have the same bisectors, show that the feet of the perpendiculars from O on the sides of the quadrilateral APDQ lie on a circle touching the nine-points circle.

382. (V. RAMASWAMI AIYAR, M.A.) :—ABC and DEF are two triangles self-conjugate with respect to a rectangular hyperbola. Prove that the pedal circles of the points A, B, C with respect to the triangle DEF and the pedal circles of the points D, E, F with respect to the triangle ABC are all coaxal.

383. (D. D. KAPADIA, M.A., B.Sc.) :—If n be a positive integer, show that—

$$\tfrac{1}{2}\sin n\alpha + \sin(n-1)\alpha\cos\theta + \sin(n-2)\alpha\cos 2\theta + \ldots + \sin\alpha\cos(n-1)\theta$$
$$= \tfrac{1}{2}\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta}\sin\alpha.$$

Hence or otherwise deduce the following results—

(i) $(n-1)\cos\theta + (n-2)\cos 2\theta + (n-3)\cos 3\theta + \ldots + \cos(n-1)\theta = \tfrac{1}{2}\left(\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2} - n\right)$;

(ii) $(n-1)^2\sin\theta + (n-2)^2\sin 2\theta + (n-3)^2\sin 3\theta + \ldots + \sin(n-1)\theta = \tfrac{1}{4}\sin\theta\operatorname{cosec}^2\frac{\theta}{2}\left(n^2 - \operatorname{cosec}^2\frac{\theta}{2}\cdot\sin^2\frac{n\theta}{2}\right)$;

(iii) $(n-1)^3\cos\theta + (n-2)^3\cos 2\theta + (n-3)^3\cos 3\theta + \ldots + \cos(n-1)\theta = \tfrac{1}{4}\left(3n^2\operatorname{cosec}^2\frac{\theta}{2} - 3\sin^2\frac{n\theta}{2}\operatorname{cosec}^4\frac{\theta}{2} - 2n^3\right)$
$+ \tfrac{1}{2}\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}$.

What is the sum of the series

$$(n-1)^4\sin\theta + (n-2)^4\sin 2\theta + \ldots + \sin(n-1)\theta.$$

384. (D. D. KAPADIA, M.A., B.Sc.):—When the expression $\dfrac{f(x)}{\{\phi(x)\}^2}$ is resolved into partial fractions, where $\phi(x)$ is the product of simple factors, show that the necessary and sufficient condition that no terms o the type $A/(x-a)$ should occur is that

$$f'(x)\phi'(x) - f(x)\phi''(x)$$

should be divisible by $\phi(x)$.

385. (Selected) :—If $s_r \equiv 1^r + 2^r + \ldots n^r$, shew that $8(s_3 + s_5 + s_7) + 1$ is a perfect square.

386. (S. RAMANUJAN) :—Shew that

$$\int_0^\infty \frac{dx}{(1+x^2)(1+n^2x^2)(1+n^4x^2)(1+n^6x^2)\ldots} = \frac{\pi}{2(1+n+n^3+n^6+n^{10}+\ldots\ldots)}$$

where the indices in the denominator are the sums of the natural numbers.

387. (S. RAMANUJAN) :—Shew that

$$\frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \frac{3}{e^{6\pi}-1} + \ldots = \frac{1}{24} - \frac{1}{8\pi}.$$